APPLIED MATHEMATICS FOR ENGINEERS
MIDTERM 1

Code : *MAT 210*
Acad.Year: *2018-19*
Semester : *Spring*
Date : *24.03.2019*
Time : *09:40*
Duration : *110 min*

Last Name:
Name:
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Student ID:
Signature:

10 QUESTIONS ON 5 PAGES
TOTAL 100 POINTS

P1. (20)	P2. (25)	P3. (15)	P4. (20)	P5. (20)		Total. (100)

1. *(10×2=20pts)* Indicate whether a given statement is **TRUE** or **FALSE** by circling your answer. No explanations are required.

TRUE / [✓ FALSE] The matrix product AB has the same number of rows as B.

TRUE / [✓ FALSE] You can compute the approximate solution to $y'' = xy + 1,\ y(1) = 1,\ y(2) = 0$ with $h = \frac{1}{2}$ using Euler's method.

[✓ TRUE] / FALSE Any solution to $A\mathbf{x} = \mathbf{b}$ is also a solution to $A^{\mathrm{T}}A\,\hat{\mathbf{x}} = A^{\mathrm{T}}\mathbf{b}$.

[✓ TRUE] / FALSE You can compute the approximate solution to $y' = xy + 1,\ y(2) = 1$ on $[-2, 2]$ with $h = 2$ using backwards Euler.

[✓ TRUE] / FALSE If $A\mathbf{x} = \mathbf{0}$ has a non-zero solution, then in the LU decomposition of A one row does not have a pivot.

TRUE / [✓ FALSE] If one column in the LU decomposition of A does not have a pivot, then $A\mathbf{x} = \mathbf{b}$ has at most one solution.

TRUE / [✓ FALSE] In the QR decomposition of A, the matrix Q is always upper triangular.

[✓ TRUE] / FALSE When dividing using LU decomposition, we first divide by L.

[✓ TRUE] / FALSE If the columns of a matrix are all perpendicular to each other then the matrix is scaled orthogonal.

[✓ TRUE] / FALSE The point-wise centered difference operator is ${y_k}' = \frac{1}{2h}\left(y_{k+1} - y_{k-1}\right)$

2. *(10pts)* Compute the approximate solution using Euler's method on the interval $[-2, 2]$ with $h = 2$

$$y' = xy - y \qquad y(2) = 1$$

(Write your answer in the form of a table of x and y values.)

use backwards Euler:

$$y_{n-1} = y_n - h y_n'$$

x_k	$y_k' = x_k y_k - y_k$	y_k
$x_0 = -2$		$y_0 = -1 - 2\cdot 1 = -3$
$x_1 = 0$	$y_1' = 0\cdot -1 - -1 = 1$	$y_1 = 1 - 2\cdot 1 = -1$
$x_2 = 2$	$y_2' = 2\cdot 1 - 1 = 1$	$y_2 = y(2) = 1$

$+h$ $+h$

Approx. sol.:

x	y_i
-2	-3
0	-1
2	1

3. *(15pts)* Discretize the following 2nd order differential equation. (Use any method.)

(A) $y'' + y' = 2x$ with $\begin{cases} y(-1) = 1 \\ y'(1) = 2 \end{cases}$ on $[-1, 1]$ with $h = 1/2$.

Pointwise Formula Method:

use $y_k'' = \dfrac{1}{\cancel{(1/2)^2}}\overset{4}{} (y_{k+1} - 2y_k + y_{k-1})$

use $y_k' = \dfrac{1}{\cancel{2\cdot 1/2}}\underset{1}{} (y_{k+1} - y_{k-1})$

x_k	y_k	$\underline{y_k'' + y_k' = x}$
$x_0 = -1$	$y_0 = y(-1) = 1$	X
$x_1 = -1/2$	y_1	$4(y_2 - 2y_1 + \overset{1}{\cancel{y_0}}) + y_2 - \overset{1}{\cancel{y_0}} = -1$
$x_2 = 0$	y_2	$4(y_3 - 2y_2 + y_1) + y_3 - y_1 = 0$
$x_3 = 1/2$	y_3	$4(\underset{y_3+1}{\cancel{y_4}} - 2y_3 + y_2) + \underset{y_3+1}{\cancel{y_4}} - y_2 = 1$
$x_4 = 1$	$y_4 = y_3 + 1$	X

$$\overset{1}{\cancel{2}} = y'(1) = y_4' = \frac{1}{1/2}(y_4 - y_3)$$

backwards diff.

simplified formulas:
(move constants to right)

$$-8y_1 + 5y_2 = -4$$

$$3y_1 - 8y_2 + 5y_3 = 0$$

$$3y_2 - 3y_3 = -4$$

$\Updownarrow$

matrix equation:

$$\begin{bmatrix} -8 & 5 & 0 \\ 3 & -8 & 5 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ -4 \end{bmatrix}$$

OR:
Using 2nd order and centered diff matrices for the fixed-free case:

$y_0 = 1$

$$\left(4\cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & \cancel{-2}\,{-1} \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 1\cdot \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & \cancel{0}\,1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - 1\cdot\left(4\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right) - 1\cdot\left(4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$+$ $+$

$y_4 = y_3 + 1$

4. *(5pts)* Find all solutions to $\begin{array}{c}① \\ ②\end{array}\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

⇗⇙ backward subst.

② : $2x + 2y - z = 2 \longrightarrow$ choose $x = s$ and $y = t \in \mathbb{R}$ freely, then

$z = 2s + 2t - 2$

① : $w + \underbrace{s}_{x} + 2\underbrace{(2s + 2t - 2)}_{z} = 0 \iff w = -5s - 4t + 4$

$$\boxed{\text{Sols: } \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5s - 4t + 4 \\ s \\ t \\ 2s + 2t - 2 \end{bmatrix} = s\begin{bmatrix} -5 \\ 1 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} -4 \\ 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 0 \\ -2 \end{bmatrix} \text{ where } s, t \in \mathbb{R}.}$$

5. *(5pts)* Find the best approximate solution to $\overbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}}^{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$

$\underline{c}_1 \quad \underline{c}_2 \qquad\qquad \underline{b}$

normal equation $A^TA\,\hat{x} = A^T b$:

$\underline{c}_1\cdot\underline{c}_1 \quad \underline{c}_1\cdot\underline{c}_2 \qquad \underline{c}_1\cdot\underline{b}$

$$\begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

$\underline{c}_2\cdot\underline{c}_2 \qquad \underline{c}_2\cdot\underline{b}$

Solve using $(A^TA)^{-1}$:

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \underbrace{\frac{1}{12-4}\begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix}}_{(A^TA)^{-1}}\begin{bmatrix} 6 \\ 10 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 16 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\boxed{\text{Best approx. sol.: } \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

6. *(5pts)* Find the best fit function $f(t) = a + bt$ to the data

t	-1	0	1
$f(t)$	-1	1	3

$a + bt = f(t)$

$a + b\cdot -1 = -1$

$a + b\cdot 0 = 1$

$a + b\cdot 1 = 3$

matrix eqn

$$\iff \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \implies$$

$\underline{c}_1 \quad \underline{c}_2 \qquad \underline{v}$

matrix is scaled orthogonal, so normal eqn has sol.:

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} 3/3 \\ 4/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\frac{\underline{v}\cdot\underline{c}_1}{\underline{c}_1\cdot\underline{c}_1} \qquad \frac{\underline{v}\cdot\underline{c}_2}{\underline{c}_2\cdot\underline{c}_2}$

Best fit function is $f(t) = 1 + 2t$.

7. *(10pts)* Use the given LU decomposition to divide. (Multiplying LU is not allowed!)

$$\underbrace{\begin{bmatrix}1&0&0\\-1&1&0\\2&-2&1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}2&1&-1\\0&1&-2\\0&0&1\end{bmatrix}}_{U}\underbrace{\begin{bmatrix}x\\y\\z\end{bmatrix}}_{\underline{x}}=\begin{bmatrix}-1\\1\\-1\end{bmatrix}$$

<u>Divide by L:</u>

$$\begin{bmatrix}1&0&0\\-1&1&0\\2&-2&1\end{bmatrix}\overbrace{\begin{bmatrix}a\\b\\c\end{bmatrix}}^{=U\underline{x}}=\begin{bmatrix}-1\\1\\-1\end{bmatrix}\underset{\text{subst.}}{\overset{\text{forward}}{\iff}}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

<u>Divide by U:</u>

$$\begin{array}{c}①\\②\\③\end{array}\begin{bmatrix}2&1&-1\\0&1&-2\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-1\\0\\1\end{bmatrix}\underset{\text{subst.}}{\overset{\text{backward}}{\iff}}\begin{cases}③:\ z=1\\②:\ y-2\cdot\underbrace{1}_{z}=0\iff y=2\\①:\ 2x+\underbrace{2}_{y}-1\cdot\underbrace{1}_{z}=-1\iff x=-1\end{cases}$$

$$\boxed{\text{Sol: }\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-1\\2\\1\end{bmatrix}}$$

8. *(10pts)* Use the given QR decomposition to find the best approximate solution. (Gaussian elimination, solving systems, or LU decomposition is not allowed!)

$$\overbrace{\begin{array}{cc}\begin{bmatrix}1&1\\-1&-1\\2&-1\\-1&0\end{bmatrix}\\ \underline{c_1}\quad\underline{c_2}\end{array}}^{Q}\overbrace{\begin{bmatrix}1&2\\0&1\end{bmatrix}}^{R}\begin{bmatrix}x\\y\end{bmatrix}=\underbrace{\begin{bmatrix}3\\0\\9\\0\end{bmatrix}}_{\underline{b}}$$

<u>Normal eqn</u> with Q (← scaled orth.!):

$\dfrac{\underline{b}\cdot\underline{c_1}}{\underline{c_1}\cdot\underline{c_1}}$ ↙

$$Q^TQ\begin{bmatrix}\hat a\\\hat b\end{bmatrix}=Q^T\begin{bmatrix}3\\0\\9\\0\end{bmatrix}\iff\begin{bmatrix}\hat a\\\hat b\end{bmatrix}=\begin{bmatrix}\frac{3+18}{7}\\\frac{3-9}{3}\end{bmatrix}=\begin{bmatrix}3\\-2\end{bmatrix}$$

↗ $\dfrac{\underline{b}\cdot\underline{c_2}}{\underline{c_2}\cdot\underline{c_2}}$

<u>Divide by R:</u>

$$\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}\hat x\\\hat y\end{bmatrix}=\begin{bmatrix}3\\-2\end{bmatrix}\underset{\text{subst.}}{\overset{\text{b.w.}}{\iff}}\begin{bmatrix}\hat x\\\hat y\end{bmatrix}=\begin{bmatrix}7\\-2\end{bmatrix}$$

$$\boxed{\text{Best approx. sol.: }\begin{bmatrix}\hat x\\\hat y\end{bmatrix}=\begin{bmatrix}7\\-2\end{bmatrix}}$$

9. *(10pts)* Compute the LU decomposition for $A = \begin{bmatrix} 2 & 3 & -2 & 1 \\ 4 & 6 & -2 & 3 \\ -6 & -9 & 2 & -2 \end{bmatrix}$

$-2x$ $+3x$ $+2x$

$$\begin{bmatrix} 2 & 3 & -2 & 1 \\ 4 & 6 & -2 & 3 \\ -6 & -9 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \boxed{2} & 1 & 0 \\ \boxed{-3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -4 & 1 \end{bmatrix}$$

undo

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & \boxed{-2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

undo

So:
$$\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}}$$

10. *(10pts)* Compute the scaled QR decomposition for $A = \begin{bmatrix} 1 & 1 & 3 \\ -1 & -1 & 0 \\ 2 & -1 & 9 \\ -1 & 0 & 0 \end{bmatrix}$

$\frac{c_2 \cdot c_1}{c_1 \cdot c_1} = \frac{0}{7} = 0$

Subtract x

$$\begin{bmatrix} 1 & 1 & 3 \\ -1 & -1 & 0 \\ 2 & -1 & 9 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 3 \\ 2 & -1 & 3 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & \boxed{0} & \boxed{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -1 & 1 \\ 2 & -1 & 1 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & \boxed{-2} \\ 0 & 0 & 1 \end{bmatrix}$$

undo undo

$\frac{c_3 \cdot c_1}{c_1 \cdot c_1} = \frac{21}{7} = 3$

subtr x $\frac{c_3 \cdot c_2}{c_2 \cdot c_2} = \frac{-6}{3} = -2$

So:
$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -1 & 1 \\ 2 & -1 & 1 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$